Categorisation of Mental Computation Strategies to Support Teaching and to Encourage Classroom Dialogue

Judy Hartnett
Brisbane Catholic Education
<jhartnett@bne.catholic.edu.au>

Mental strategies are a desired focus for computational instruction in schools and have been the focus of many syllabus documents and research papers. Teachers though, have been slow to adopt such changes in their classroom planning. A possible block to adoption of this approach is their lack of knowledge about possible computation strategies and a lack of a clear organisation of a school program for this end. This paper discusses a framework for the categorisation of mental computation strategies that can support teachers to make the pedagogical shift to use of mental strategies by providing a framework for the development of school and classroom programs and provide a common language for teachers and students to discuss strategies in use.

Mental computation has been the focus of a major shift in mathematics education in many parts of the world. Recent curriculum documents in Australia and overseas the United States *Principles and Standards for School Mathematics* (National Council of Teachers of Mathematics, 2000), the new United Kingdom *Primary Framework for Literacy and Mathematics* (DfES, 2007), the Dutch *Specimen of a National Program for Primary Mathematics* (Treffers & DeMoor, 1990), and the Australian *National Statement on Mathematics for Australian Schools* (Australian Education Council, 1991) have indicated that mathematics education needs to change emphasis to match the developments in the world today.

Syllabus documents in all states of Australia advise teachers to take an approach focusing more on mental computation as part of a range of strategies and less on traditional written algorithms. For example, the Level 2 Addition and Subtraction outcome in the Queensland Studies Authority Years 1-10 Mathematics syllabus (2004) states: "Students identify and solve addition and subtraction problems involving whole numbers, selecting from a range of computation methods, strategies and known number facts" (p. 19). The benefits of a focus on mental computation have been widely reported and include the need for school mathematics to be useful and to reflect computational techniques used in everyday life (Australian Education Council, 1991; Clarke, 2003; Irons, 2000; Willis, 1990; Zevenbergen, 2000).

Mental computation strategies are different from written algorithms in that they require more than the application of a remembered procedure. The key difference is the need for some application of a deeper knowledge of how numbers work. Callingham (2005) discussed research in mental computation as focussing on "identifying and describing students' strategies for addressing particular kinds of calculations, often within a framework of number sense" (p. 193). Number sense has been defined as having a "general understanding of number and operations along with an ability and inclination to use this understanding in flexible ways" (McIntosh, Reys, Reys, Bana, & Farrell, 1997, p. 3). Using mental computation strategies flexibly requires sound number sense and by using a strategies approach to computation, rather than a focus on procedural algorithms, students have opportunities to work with numbers in flexible ways, which in turn, provide opportunities for them to improve their number sense. Needing number sense for efficient use of computation strategies, and the development of number sense by using such strategies, are very closely interrelated.

Mental Computation Strategies

There has been discussion in the literature of what constitutes a mental computation strategy. Earlier definitions of mental computation focussed on the lack of written recordings. Trafton (1978) described the use of non standard algorithms for the computation of exact answers without the use of pencil and paper. Sowder (1988) defined mental computation as "the process of carrying out arithmetic calculations without the aid of external devices" (p. 182). Threlfall (2002) described strategies, as "where students can be correct by constructing a sequence of transformations of a number problem to arrive at a solution as opposed to just knowing, simply counting or making a mental representation of a 'paper and pencil' method" (p. 30). The Queensland Years 1-10 Mathematics syllabus (Queensland Studies Authority, 2004) provides examples of mental computation strategies in early levels such as, "count on and back, doubles, make to ten" (p. 45) and in later levels "making numbers manageable" (p. 46). Some of these "strategies", for example, "turnarounds (commutativity)" are not strategies as thought process as discussed above, but are skills more related to having sound number sense. These understandings would be used as part of a strategy (i.e., a sequence of transformations of a number problem) to solve a problem but are difficult to consider as strategies themselves.

Strategy Categorisation

In research literature there have been many attempts to describe lists of possible mental computation strategies. A well documented strategy categorisation by Beishuizen (1985) described two main strategies for mental addition and subtraction. The strategy 1010 referred to splitting numbers into tens and ones and dealing with the parts separately, left to right. N10 referred to a strategy where one number is split into tens and ones and the tens of the second number are added to the first number followed by the ones. Many authors refer to these as the two main strategies for addition and subtraction of numbers to 100 (Cobb, 1995; Cooper, Heirdsfield, & Irons, 1996; Fuson, 1992; Reys, Reys, Nohda, & Emori, 1995; Thompson, 1994). Beishuizen, Van Putten, and Van Mulken (1997) extended this list to include a strategy they referred to as A10, where the second number is split to facilitate a bridge to a multiple of ten and then the remainder is added to the first number. This dealt with problems that required bridging of a ten in either addition or subtraction. A further paper by Klein, Beishuizen, and Treffers (1998) discussed another strategy that they called N10C, where the second number is rounded up to a multiple of ten and this number is added to the first number followed by an adjustment or compensation for the rounding. Yackel (2001) described "collections-based" solutions where both numbers are broken into parts, usually tens and ones (compare to 1010), and "counting or sequence based" solutions starting with one number and dealing with the others progressively, part by part (compare to N10).

Cooper, Heirdsfield, and Irons (1996) developed a strategy schema based on work of Beishuizen (1993) to analyse strategies used in a study of young children's mental addition and subtraction accuracy and strategy usage. Their schema consisted of four strategy categories: i) Counting, ii) Separation (1010) which they further categorised to be right to left, left to right or cumulative, iii) Aggregation (N10), again categorised further as right to left or left to right, and iv) Wholistic, which described strategies involving adjustment of one number by compensation (N10C) or by levelling where both numbers were adjusted to create a new equivalent question. They also included a separate category for students who reported using a mental image of the pen-and-paper algorithm.

Often lists of strategies have been derived from studies where computation problems were presented to students and the strategies that the students actually exhibited were analysed and categories emerged. For example, Reys, Reys, Nohda, and Emori (1995), used a mental computation test in their study of the performance and strategy use of students in Japan. Prior to administering the test the researchers formulated a detailed categorisation of anticipated strategies. Their categorisation reflected similar major grouping as described above and used letters to identify the major strategies and then variations of these strategies were numbered e.g. A1, A2, B1, etc. The categories labelled A involved grouping of tens and ones separately (compare to 1010), those labelled B had one number held constant (compare to N10), and those labelled C involved rounding of one or both numbers to multiples of ten (compare to N10C).

Wigley (1996) described strategies for addition and subtraction where numbers were split and recombined in different ways using knowledge of place value and complementation, which he described as an ability to generate relationships associated with complements in numbers to ten or hundred. He advocated teaching strategies for multiplication that used doubling and halving, including repeated doubling and halving, and the trial and use of multiplication and subtraction to achieve progressively smaller remainders as a strategy for division.

Teaching Mental Computation Strategies

In the literature two different approaches to the teaching of mental computation strategies are described. One focuses on students inventing or using their own intuitive strategies to solve given computation problems (e.g. Buzeika, 1999; Heirdsfield, 2004, 2006) and others describe where particular strategies were the focus of teaching (e.g., Beishuizen, 1999). In all of these studies and others (Buys, 2001; Beishuizen, 2001) students were encouraged to discuss strategies used.

Threlfall (2002) argued that a teaching approach that is intended to foster choice and flexibility by teaching wholistic strategies needs to be underpinned by a coherent way of thinking about the possible choices, "so that they can be taught in an organised and systematic way. In other words, there has to be a categorisation system that makes sense to the teacher" (p. 32). He was concerned that an incomplete set of strategies may lead to efficient strategies not being available for use because they had not been taught. Mental arithmetic needs to be taught using methods quite different from traditional pencil-and-paper methods. Offering only one method is too rigid. Leaving pupils to find their own methods will deprive many of more advanced strategies (Wigley, 1996).

Many teachers in classrooms today were students themselves in a period when mathematics teaching focussed on rote learning of basic facts and on the development of procedures for "successful" completion of traditional written algorithms. These teachers consciously know of very few if any computation strategies other than the use of vertical algorithms in the mind. Although these teachers can see benefits for including mental computation strategies in their teaching programs their lack of knowledge leads to a lack of confidence and lack of teaching ideas to take the idea forward into their practice. If a comprehensive but easy to understand list of possible strategies were organised based on the research in this area a useful tool to change classroom pedagogy and therefore improvement of student learning outcomes could be achieved.

The Mental Computation Strategy Framework

The author of this paper has attempted to create a categorisation framework for the purpose of informing and providing structure for the *teaching* of computation strategies. The intention of the strategy categorisation was to create a small number of general categories with intuitive labels using simple language that would make sense to teachers and also to students. Then a list of sub-categories would make clearer the variations that could be a focus in each category. In all, five major categories and twenty-one sub-categories were identified. It was also an intention that these categories would be applied across the range of the primary school year levels at least, and across the four operations with whole numbers, common and decimal fractions, negative numbers, as appropriate. This way a school could utilise the framework for a whole school program or approach to the teaching of mental computation strategies. With the labels for the categories kept in simple intuitive language it was intended that these names would be used in the classroom as an aid the discussion of strategies used by students and as part of lessons on particular strategies. It is a coherent way of thinking about the possible mental computation strategies that the researcher is interested in providing to meet an identified need from teachers and schools.

A description of the categories and links to other categorisations in the literature are outlined in Table 1. The intention was not to find a single description for each possible strategy but to provide a framework for teachers to base their development of programs of lessons on and for teachers and students to use as a common language to describe ways of working through computation examples.

Method

The focus class consisted of 27 Year 3 students who were approximately 8 years of age in a suburban school in Brisbane, Queensland. There was a wide range of abilities within this class and the teacher was experienced and had taught this year level for many years. Year 3 was chosen for the study as traditionally addition and subtraction algorithms were introduced in this year of schooling. The teacher was interested in the inclusion of mental strategies into the class number program. She perceived there would be benefits for the class by shifting the focus away from the algorithm to the development of mental computation strategies and she was prepared to put teaching of algorithms aside for the whole year.

The class number program was planned to introduce and focus teach one major strategy category from the framework each school term. “Counting On and Back” was the focus in first term, followed by “Breaking Up numbers” in term 2, “Adjusting and Compensating” (also called change and fix especially when working with the students) in term 3 leaving “Doubling and Halving” for fourth term, which linked to other planned focus work on multiplication and division. The “Use Place Value” category was not a particular focus for

Table 1

Categorisation of Mental Computation Strategies and Links to Literature

	Related categorisations and References
Count On and Back:	
Count on to add	Counting *(Cooper et al., 1996)*
	Count on or back *(McIntosh & Dole, 2005)*
Count back to subtract	Counting *(Cooper et al., 1996)*
Count on to subtract	Aggregation (additive) (*Cooper et al., 1996)*
	A10 *(Beishuizen et al., 1993, 1997)*
Count on to multiply	
Adjust and Compensate: (Change and Fix)	
Adjust one number and compensate	N10C *(Beishuizen et al., 1993, 1997)*
	C1, C2 *(Reys et al., 1995)*
	Wholistic compensation *(Cooper et al., 1996)*
	Over jump method (*Thompson, 1999*)
Adjust two numbers and compensate	C3, D1 *(Reys et al., 1995)*
Adjust two numbers	Wholistic levelling *(Cooper et al., (1996)*
Double and /or Halve:	
Use a double or near double to add or subtract	Doubles / near doubles *(McIntosh & Dole, 2005)*
Double to multiply by 2	
Double, double to multiply by 4	
Double, double, double to multiply by 8	Repeated doubling *(Wigley, 1996)*
Half to divide by 2	
Half, half to divide by 4	
Half, half, half to divide by 8	Repeated halving *(Wigley, 1996)*
Double and halve	
Break Up Numbers:	
Break up two numbers using place value	1010 *(Beishuizen et al., 1993, 1997)*
	A1, A3 *(Reys et al., 1995)*
	Separation *(Cooper et al., (1996)*
	Split method (*Thompson, 1999*)
	Split tens method *(McIntosh & Dole, 2005)*
Break up two numbers using compatible nos.	Split jump method *(Thompson, 1999)*
Break up one number using place value	N10 *(Beishuizen et al., 1993, 1997)*
	B1, B2 *(Reys et al., 1995)*
	Aggregation (Cooper et al., 1996)
	Jump method (*Thompson, 1999*)
	Sequential method *(McIntosh & Dole, 2005)*
Break up one number using compatible nos.	A10 *(Beishuizen, Van Putten, & Van Mulken, 1997)*
Use Place Value:	
Think in multiples of ten	
Focus on relevant places	

any term as it is limited to particular problems and was simply introduced where appropriate.

Throughout all instruction and practise activities students were encouraged to show their thinking using any written methods they felt comfortable with. The classroom climate also encouraged discussion and flexibility of choice of strategy. The students completed practise activities for each strategy but when given open computation problems to solve were free to use any strategy they liked. A range of models to support the learning were used throughout the year which included ten frames, numbered lines, open number lines, and number boards.

The students were given a pre-test, mid year test, and post test in which they were asked to complete the computations and show what they were thinking and how they worked out each question. The items were chosen to present addition or subtraction situations that could be solved using some of the strategies they would be taught throughout the year. The items were presented as single computations presented horizontally without context. The intention was to keep the questions as clear and free of distractions as possible. The students were not interviewed, as previous studies, including one quoted in Threlfall (2002), found that written responses attained when students were asked to "work out each answer mentally and write down how they had done it" (p. 33) took the same form as the protocol responses. An aim of the study was to look for evidence of strategy categories in the written responses of the students across the year.

Results and Discussion

The use of the four main strategy categories from the framework as the basic focus of instruction for each of the four terms of the year made sense to the teacher and the students and was an effective program organiser. The teacher was interviewed and stated that this organisation was easy to follow and gave her confidence to teach the strategies. The teacher saw it as clarifying and observed that the students were generally comfortable with the strategies by the end of each term of learning. The students exhibited a growing repertoire of strategies as the year progressed and showed an early ability to use a variety of strategies, evidenced by growth in the number of strategies used for the pre to post tests (See Table 2). The lack of obvious use of strategies did not mean the students did not use strategies but just that they chose not to or, more likely, lacked confidence or methods to record these.

Table 2

Number of Students who used a Variety of Different Strategies

	Pre test	Post test
0 strategies evident	21	0
1 strategy	5	3
2 strategies	3	6
3 strategies	0	4
4 strategies	0	4
5 strategies	0	4
> 5 strategies	0	7

In the mid year and post tests particularly, evidence of the students' use of the strategies in the working and descriptions of the way they solved the problems showed strategies named specifically using the framework. Figure 1 shows four examples of such responses.

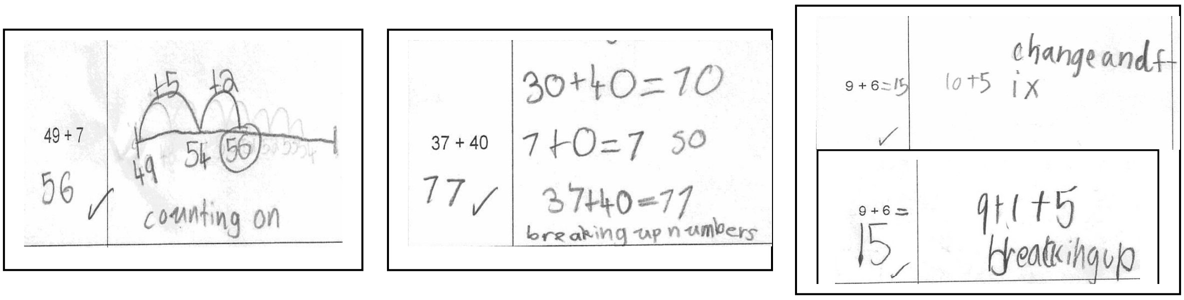

Figure 1: Student work samples showing use of the strategy categorisation framework.

There was also a variation between strategies used by the same students on different instruments. One instrument inadvertently was given to the students by the researcher and again by the class teacher one week apart. There was a large number of students who used a completely different strategy on the same item on each test.

Conclusion

This study was only for one year and was in a year early in primary school. For the framework to be evaluated, a longer period of sustained use for teaching and learning is required. Further monitoring is required on using this framework to plan a whole school program across all year levels, all types of numbers (ie., including decimals, common fractions, etc) and across all operations. The focus school is currently using this framework to do just this with the assistance of the researcher.

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